1. (20 points) Quick sort.

a. (5 points) Study the pseudocode of the Partition algorithm in slide set 7-qsort.ppt. Using Slide #11 as a model, illustrate the operation of Partition on array \( A = [14 \ 1 \ 16 \ 11 \ 12 \ 20 \ 4 \ 15 \ 3 \ 19] \). Importantly, indicate where is the pivot element when the algorithm terminates.

b. (5 points) Using Slide #12 as a model, illustrate the operation of Quick Sort on array \( A \) above.

c. (5 points) The Partition algorithm (slide #10) may have some problem when all elements are equal. Try to apply Partition on array \( D = [5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5] \) and discuss the problem. What will be the running time of quick sort for this kind of array?

d. (5 points) In the pseudocode for Quicksort on Slide #5, Quicksort was recursively applied to two subarrays: one subarray from index \( p \) to \( q - 1 \), and the other from index \( q + 1 \) to \( r \). This is correct because element \( A[q] \) is already in the correct position. Now consider the following modified pseudocode for Quicksort:

\[
\text{QUICKSORT}(A, p, r) \\
\quad \text{if} \ (p < r) \\
\quad \quad \text{then} \ q \leftarrow \text{PARTITION}(A, p, r) \\
\quad \quad \text{QUICKSORT}(A, p, q - 1) \\
\quad \quad \text{QUICKSORT}(A, q, r)
\]

The only difference between the pseudocode above and the original one is the last line. In the above pseudocode, the second subarray starts from \( q \) instead of \( q + 1 \). Besides having to sort one extra element, this modified algorithm will sometimes fail to terminate. Give an example array for which the modified algorithm will fail to terminate.

e. (5 points) How many times was the procedure Partition called in the two examples above? (The answer can also help you solve the next problem.)

2. (15 points) Analysis of Randomized Quick Sort.

During lecture, we analyzed Randomized Quick Sort and proved its expected running time using the substitution method. This homework problem concerns the number of times that the procedure Partition is called.

a. (6 points) Using the analysis on Slide #33 as a template, write down a recurrence for the expected number of times that the procedure Partition is called in Randomized Quick Sort, for an input of size \( n \).

b. (7 points) Using the substitution method to prove that the expected number of calls to Partition is \( O(n) \).

c. (2 points) Provide an intuitive explanation why there is \( O(n) \) calls to Partition. (Hint: after each call to Partition, how many elements will be sorted into the right position?)

3. (15 points) Analysis of a randomized algorithm.

During lecture, we analyzed Randomized Quick Sort and proved its expected running time using the substitution method. Here we practice the analysis with a simple randomized algorithm.
AlgR(A[1..n])
if (n <= 1) return;
Func(A[1..n]); // Func has a running time of Θ(n).
r = rand(); // Let r be a random number between 0 and 1;
if (r < 0.5) { // with probability 0.5
    AlgR(A[1..n/2]); // AlgR is applied to first subarray of size n/2
    AlgR(A[n/2+1..n]); // AlgR is applied to second subarray of size n/2
} else { // with probability 1 - 0.5
    AlgR(A[1..n/3]); // AlgR is applied to a subarray of size n/3
}

a. (3 points) What is the best case running time of RandAlg as a function of n?
b. (3 points) What is the worst case running time of RandAlg as a function of n?
c. (4 points) Using the analysis on Slide #31 as a template, write down a recurrence for the expected running time of AlgR as a function of n.
d. (5 points) Using substitution method to prove that the expected running time of AlgR is O(n).

4. (10 points) Indicate whether the following statement is true or false. Briefly justify your answers.
a. Quick sort runs in linear time for an already sorted array.
b. Quick sort runs in Θ(n^2) time in the worst case.
c. Randomized quick sort runs in Θ(n log n) time in the worst case.
d. The expected running time is Θ(n log n) for both quick sort and randomized quick sort.

5. (30 points) Heap and heap sort.
a. (3 points) Is the sequence [23, 17, 14, 6, 13, 10, 1, 5, 7, 12] a max-heap?
b. (8 points) Study the pseudocode of Heapify and BuildHeap on Slide #16 and #29 in lecture10.ppt. Using Slides #30 – 42 as a model, illustrate the operation of BuildHeap on array A = [14 15 10 5 6 16 11 3 12 20]. Figure 1 on page 3 is provided for your convenience. You only need to show the content of the tree each time a node is heapified. Make sure your final tree is indeed a heap.
c. (8 points) Starting from the heap shown in Figure 2 (page 4), show the content of the new heap after each heap operation.
d. (5 points) Why do we want the loop index i in algorithm BuildHeap to decrease from ⌊length(A)/2⌋ to 1 rather than increase from 1 to ⌊length(A)/2⌋?
e. (6 points) What is the running time of heapsort on an array A of length n that is already sorted in increasing order? What about decreasing order? (Hint: first think about the cost for building heap in these two cases, then think about the cost for the actual sorting part. You can use examples [0 1 2 3 4 5 6 7 8 9] or [9 8 7 6 5 4 3 2 1 0] to help you think.)

6. (15 points) Building a heap using insertion.
The procedure BuildHeap can also be implemented by repeatedly using HeapInsert to insert the elements into the heap. Consider the following implementation:
BuildHeap2(A)
    heapsize(A) = 1;
    for (i = 2 to length(A))
        HeapInsert(A, A[i]);

a. (8 points) illustrate the operation of BuildHeap2 on array A = [14 15 10 5 6 16 11 3 12 20]. Show the content of the heap and the array A after each HeapInsert. Compare the result with the result obtained above in 4c. Do the two procedures create the same heap at the end?

b. (5 points) What is the worst-case time complexity of BuildHeap2? Briefly justify your answer.

c. (2 points) How does the running time of the two procedures compare?

7. (Extra credit: 15 points) Nuts and bolts. You are given a collection of n bolts of different widths and n corresponding nuts. You are allowed to try a nut and bolt together, from which you can determine whether the nut is larger than the bolt, smaller than the bolt, or matches the bolt exactly. However, there is no way to compare two nuts together or two bolts together. The problem is to match each bolt to its nut. Design an efficient algorithm for this problem with average-case efficiency (or equivalently, expected running time) in \( \theta(n \log n) \).
Figure 1: Build heap
Figure 2: Heap operations