1. Order of growth (20 points)

Order the following functions according to their order of growth from the lowest to the highest. If you think that two functions are of the same order (i.e \( f(n) \in \Theta(g(n)) \)), put them in the same group.

\[
\log \log n, \log n, \log(n!), n^{1.1}, n \log(n), \sqrt{n}, n^2, n!, (n + 1)!, 2^n, 3^n, 2^{n+1}.
\]

2. Asymptotic Notation (20 points)

For each pair of functions in the table below, determine whether \( f(n) \in O(g(n)) \), \( f(n) \in \Omega(g(n)) \), or \( f(n) \in \Theta(g(n)) \). Complete the table following the examples in the first three lines.

<table>
<thead>
<tr>
<th></th>
<th>( f(n) )</th>
<th>( O )</th>
<th>( \Omega )</th>
<th>( \Theta )</th>
<th>( g(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i.</td>
<td>( n^2 )</td>
<td>( X )</td>
<td></td>
<td></td>
<td>( n^4 )</td>
</tr>
<tr>
<td>ii.</td>
<td>( n )</td>
<td>( X )</td>
<td></td>
<td>( \log n )</td>
<td></td>
</tr>
<tr>
<td>iii.</td>
<td>( n^2 - n )</td>
<td>( X )</td>
<td>( X )</td>
<td>( 5n^2 + 6n )</td>
<td></td>
</tr>
<tr>
<td>a.</td>
<td>( n^2 + 2n )</td>
<td>( X )</td>
<td>( X )</td>
<td></td>
<td>( 9n )</td>
</tr>
<tr>
<td>b.</td>
<td>( n + 2\sqrt{n} )</td>
<td></td>
<td></td>
<td></td>
<td>( 2n )</td>
</tr>
<tr>
<td>c.</td>
<td>( n \log n )</td>
<td></td>
<td></td>
<td></td>
<td>( n\sqrt{n} )</td>
</tr>
<tr>
<td>d.</td>
<td>( n + \log n )</td>
<td></td>
<td></td>
<td></td>
<td>( \sqrt{10n} )</td>
</tr>
<tr>
<td>e.</td>
<td>( 3n^2 + 4n + 1 )</td>
<td></td>
<td></td>
<td></td>
<td>( n^n )</td>
</tr>
<tr>
<td>f.</td>
<td>( 2^n )</td>
<td></td>
<td></td>
<td></td>
<td>( 3^{n/2} )</td>
</tr>
<tr>
<td>g.</td>
<td>( \log(n^4) )</td>
<td></td>
<td></td>
<td></td>
<td>( n \log n )</td>
</tr>
<tr>
<td>h.</td>
<td>( 2\log^2 n )</td>
<td></td>
<td></td>
<td></td>
<td>( \log(n + 1) )</td>
</tr>
<tr>
<td>i.</td>
<td>( \log(n!) )</td>
<td></td>
<td></td>
<td></td>
<td>( \log(n^n) )</td>
</tr>
<tr>
<td>j.</td>
<td>( \log_{10}(3n + 1) )</td>
<td></td>
<td></td>
<td></td>
<td>( \log_2(n + 5) )</td>
</tr>
</tbody>
</table>

3. Prove asymptotic notation by definition (30 points)

Using the basic definition of \( O \), \( \Omega \), and \( \Theta \), show that

(a) \( 10n^2 + 2n + 1 \in O(n^2) \)
(b) \( n^2 - 9n + 5 \in \Omega(n) \)
(c) \( 3n + 5\sqrt{n} + 2 \in \Theta(n) \).
(d) Let \( f(n) \) and \( g(n) \) be asymptotically positive functions. Prove or disprove each of the following conjectures. (To prove, use definitions of asymptotic notations. To disprove, find a counter example.)

i. \( f(n) \in O(g(n)) \) implies that \( g(n) \in O(f(n) + g(n)) \).
ii. \( f(n) \in O(g(n)) \) implies that \( g(n) \in \Omega(f(n) + g(n)) \).
iii. \( f(n) \in \Omega(g(n)) \) implies that \( 2f(n) \in \Omega(2g(n)) \).
4. Bubblesort (20 points).

Bubblesort is a sorting algorithm that works by repeatedly swapping adjacent elements that are out of order. Consider the version of Bubblesort below which sorts the array $A[1 \ldots n]$ into increasing order by repeatedly bubbling the minimum element of the remaining array to the left.

**Algorithm 1** bubblesort(int $A[1 \ldots n]$)

- $i = 1$
- **while** $i <= n$ **do**
  - $j = n$
  - //The inner while loop moves the smallest element in $A[i \ldots n]$ to $A[i]$
  - **while** ($j > i + 1$) **do**
    - end if
    - $j -= 1$
  - end while
- $i += 1$
- **end while

(a) Consider running the above sort on the array $[4, 5, 3, 1, 2]$. Show the sequence of changes which the algorithm makes to the array.

(b) Give a loop invariant for the outer while loop that will allow you to prove the correctness of the algorithm.
   (Hint: consider what is true about the first $i - 1$ entries of the array each time you touch the top of the while loop)

(c) Use induction to prove your loop invariant is true and then use this to prove the correctness of the algorithm. Specifically complete the following:
   i. Base case (Initialization)
   ii. Inductive step (Maintenance)
   iii. Termination step

(d) Give the best-case and worst-case runtimes of this sort in asymptotic (i.e., $O$) notation.